A Study of the Nonorthogonal FDTD Method Versus the Conventional FDTD Technique for Computing Resonant Frequencies of Cylindrical Cavities

Paul H. Harms, *Member, IEEE*, Jin-Fa Lee, *Member, IEEE*, and Raj Mittra, *Fellow, IEEE*

Abstract—**In this paper, the nonorthogonal finite difference time domain (FDTD) technique is used to compute the resonant frequencies of dielectric-filled cylindrical cavities. Because the method is based on the nonorthogonal coordinate system, it is not restricted to specific geometries, e.g., rectangular or axially symmetric geometries and is suitable for analyzing cavities of arbitrary shape. The advantages of this technique over the conventional FDTD algorithm with a staircase grid are readily shown in a convergence study, where the two methods are used to compute the dominant resonant frequency of a cylindrical cavity. The accuracy of the technique for calculating the resonant frequencies of the first few modes is demonstrated by comparing the results obtained via this technique with those derived by using two versions of the finite element method in the frequency domain.**

I. Introduction

ACCURATE analysis of the resonant frequencies of dielectric-filled cavities is an important area of study since such cavities are used in many microwave applications, e.g., filter and oscillator design as well as characterization of dielectric materials. By placing the material in question in a cavity and measuring the resonant frequencies and the Q factors, the complex permittivity can be obtained. For these and other applications, it would be very useful to be able to analyze cavities of arbitrary shape with arbitrary fillings. This paper proposes to use the nonorthogonal-FDTD method for the analysis of cavities of arbitrary shape and applies the method to the analysis of dielectric filled cavities [1]–[4] as a demonstration of the technique.

The nonorthogonal-FDTD method has been successfully applied to the analysis of two-dimensional electromagnetic wave scattering and three-dimensional waveguide discontinuities [1]–[3]. It is a very general method, capable of analyzing arbitrarily-shaped structures and materials with inhomogeneities, anisotropy, nonlinearities and losses. Since it is based on the nonorthogonal coordinate system, the grid can be chosen to conform to the geometry of the problem, and its density can be varied as necessary. This in turn, has the potential of saving much computer memory and time over the conventional FDTD algorithm [5] which requires that the grid retain its rectangular, orthogonal structure and that the cell density be essentially uniform throughout the entire computational domain [1], [2], [5], [6]. These restrictions create problems when applying the conventional FDTD technique to structures with curved boundaries because either the grid must be carefully modified or the staircase approximation must be incorporated [3], [6]. In addition, the grid cell size must be compatible with small but significant features of the structure under consideration like the tuning screws of resonators. These two difficulties prohibit the application of the conventional FDTD technique to many structures of interest because of the resultant dense mesh which requires extensive computer memory and the corresponding small time step that can cause the algorithm to be highly time-consuming.

The order of presentation in this paper is as follows. Section II outlines the theory of the nonorthogonal FDTD which has been included primarily for completeness, since more detail is available elsewhere [1]–[4]. Section III presents a convergence study in which the staircase-grid FDTD is compared with the nonorthogonal-grid FDTD for computing the resonant frequencies of cylindrical cavities. It will be shown that the results from the nonorthogonal grid converge much faster than those from the staircase grid. Section IV shows some results of the resonant frequency computation of two different types of dielectric-loaded cylindrical cavities and compares them both with those published in the literature as well as derived by using an FEM code.

II. Theory

A nonorthogonal FDTD algorithm [1], [2] in combination with the FFT [7] was used to compute the resonant frequencies of cylindrical cavities. For a good discussion on nonorthogonal coordinate systems consult references [8], [9]. The nonorthogonal FDTD algorithm was formulated using the covariant and contravariant components of the electric and magnetic fields as the unknown variables [1], [2]. In this technique, the integral form of Maxwell's Equations are discretized over the nonorthogonal unit cell shown in Fig. 1. The resulting differ-

Manuscript received June 13, 1991; revised November 6, 1991.

J.-F. Lee was with the Electromagnetic Communication Laboratory, Department of Electrical and Computer Engineering, University of Illinois. He is presently with the Department of Electrical Engineering, Worcester Polytechnic Institute, Worcester, MA 01609.

P. H. Harms and R. Mittra are with the Electromagnetic Communication Laboratory, Department of Electrical and Computer Engineering, University of Illinois, 1406 W. Green St., Urbana, IL 61801-2991.

IEEE Log Number 9106056.

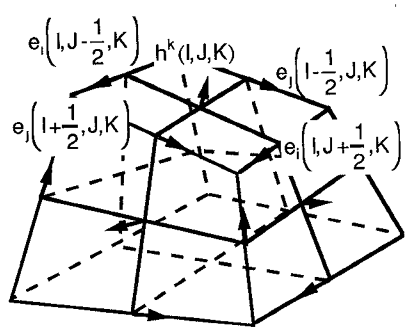

Fig. 1. Typical unit cell for the nonorthogonal lattice.

ence equations for the electric and magnetic fields are

$$\begin{aligned} e^k(I, J)^{n+1} = e^k(I, J)^n + \frac{\Delta t}{\epsilon \sqrt{g}} \Bigg(& h_j^{n+(1/2)} \left(I + \frac{1}{2}, J\right) \\ & - h_j^{n+(1/2)} \left(I - \frac{1}{2}, J\right) \\ & + h_i^{n+(1/2)} \left(I, J - \frac{1}{2}\right) \\ & - h_i^{n+(1/2)} \left(I, J + \frac{1}{2}\right)\Bigg) \end{aligned} \tag{1}$$

$$\begin{aligned} h^k(I, J)^{n+(1/2)} = h^k(I, J)^{n-(1/2)} - \frac{\Delta t}{\mu \sqrt{g}} & \\ \cdot \Bigg(e_j^n \left(I + \frac{1}{2}, J\right) - e_j^n \left(I - \frac{1}{2}, J\right) & \\ + e_i^n \left(I, J - \frac{1}{2}\right) - e_i^n \left(I, J + \frac{1}{2}\right)\Bigg) & \end{aligned} \tag{2}$$

where the i and j components of the electric and magnetic fields are given by index permutation. In addition to (1) and (2), the covariant components need to be computed from the corresponding contravariant components. For example, the covariant component, e_k, is obtained from the contravariant components of $\vec{E}$ by the equation

$$\begin{aligned} & e_k(I, J, K) \\ & = g_{33} e^k(I, J, K) + \frac{g_{31}}{4} \Bigg[e^i \left(I - \frac{1}{2}, J, K - \frac{1}{2}\right) \\ & + e^i \left(I - \frac{1}{2}, J, K + \frac{1}{2}\right) + e^i \left(I + \frac{1}{2}, J, K - \frac{1}{2}\right) \\ & + e^i \left(I + \frac{1}{2}, J, K + \frac{1}{2}\right)\Bigg] \\ & + \frac{g_{32}}{4} \Bigg[e^j \left(I, J + \frac{1}{2}, K - \frac{1}{2}\right) \\ & + e^j \left(I, J + \frac{1}{2}, K + \frac{1}{2}\right) + e^j \left(I, J - \frac{1}{2}, K - \frac{1}{2}\right) \\ & + e^j \left(I, J - \frac{1}{2}, K + \frac{1}{2}\right)\Bigg] \end{aligned} \tag{3}$$

where g_{lm} are the metrical coefficients. Similar expressions exist for the i and j covariant components of $\vec{E}$ and the i, j and k covariant components of $\vec{H}$ [1]. With this formulation, the number of computations are approximately twice that of the Cartesian based FDTD, due to the need to convert back and forth between the contravariant and covariant components. However, it is anticipated that the ability to conform the mesh to the geometry of the problem will offset this additional computational time when the technique is applied to complicated structures.

To accurately model the geometry of the resonator, the nonorthogonal mesh must be molded to the particular structure under study. This requires a flexible mesh generator, capable of producing a lattice made of arbitrarily shaped unit cells as shown in Fig. 1. We are experimenting with the commercial mesh generator, PATRAN [10], for constructing suitable grids for the nonorthogonal FDTD algorithm. For the purposes of this work, however, we wrote a computer program for producing nonorthogonal meshes of the cylindrical cavities.

For stability, the time step must satisfy the criterion [2]:

$$\Delta t \leq \frac{1}{c} \frac{1}{\sqrt{\sum_{l,m=1}^{3} g^{lm}}} \tag{4}$$

where

$$g^{lm} = \vec{a}^l \cdot \vec{a}^m \tag{5}$$

III. Convergence Study

To determine whether the nonorthogonal FDTD is actually an improvement over the conventional FDTD algorithm for modeling curved structures, a convergence study was performed. From the theory of finite-difference equations, it is well-known that the convergence of the solution of the central-differencing scheme, which is employed in the present work, is on the order of h^2, where h is the dimension of a cell in a uniform rectangular mesh [11]. For a rectangular structure, the nonorthogonal grid is equivalent to the conventional orthogonal mesh so the rate of convergence will be the same. However, for a nonrectangular object, the nonorthogonal grid conforms to the structure, and the convergence rate is significantly better than that obtainable with the conventional staircase-FDTD technique for the same structure.

To compare the convergence rate of the nonorthogonal FDTD with that of the staircase FDTD, the dominant resonant frequency of an empty cylindrical cavity was computed. A cavity 1 m. in height with a 1 m. radius was used to avoid the occurrence of any degenerate modes at the fundamental resonant frequency, which was 0.1149 GHz. For the convergence study, the cylindrical cavity was initially meshed with the coarse nonorthogonal grid as shown in Fig. 2. The cavity was excited with Gaussian pulses placed at three adjacent electric field components.

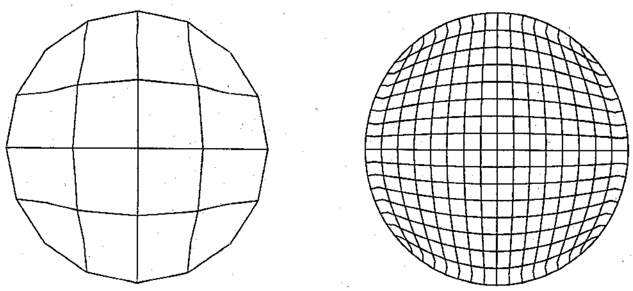

Fig. 2. Sparse and dense nonorthogonal grids for the cylindrical cavity.

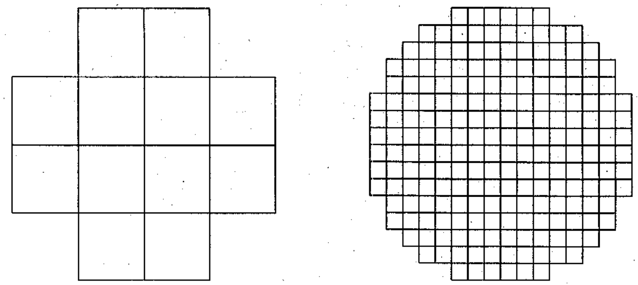

Fig. 3. Sparse and dense staircase grids for the cylindrical cavity.

Three components of the electric field were sampled, and then the FFT was taken of the time domain field samples, and the dominant resonant frequency was obtained from the result. (Since the dominant mode is the only one that we were interested in, we could have just excited and observed the axial component of the electric field.) The nonorthogonal mesh was refined several times, and the dominant resonant frequency computed for each case. The most coarse and most dense grids are shown in Fig. 2. A similar process was used with the conventional FDTD method to compute the dominant resonant frequency of the cavity, the only difference being that a staircase approximation of the cylindrical cavity was used. The most coarse and most dense staircase grids are shown in Fig. 3.

Fig. 4 shows the results of the study. The logarithm of the error is plotted as the logarithm of the cell size, h, since the error for the staircase approximation is proportional to a power of h. The error is the difference between the analytical and respective numerical values of the dominant resonant frequency, and it is normalized to the analytical resonant frequency. We note from the graph in Fig. 4 that the staircasing FDTD algorithm yields a linear curve for the logarithm of the error versus the logarithm of the cell size. As the mesh density increases, the error in the result decreases approximately as $h^{1.5}$. The power of h is less than 2 because the staircased grid is being used to model a non-rectangular geometry. On the other hand, the nonorthogonal-FDTD method converges in a nonlinear fashion. Not only is the convergence error of the nonorthogonal-FDTD mesh lower than that of the conventional FDTD mesh, but its rate of convergence increases with increasing mesh density. The nonorthogonal FDTD is significantly more accurate for equivalent mesh densities than the uniform FDTD in computing the resonant frequencies of cylindrical cavities. Even though the algorithm takes more time and memory than the conventional FDTD algorithm, its significantly greater rate of convergence over the conventional FDTD for non-rectangular structures can offset this disadvantage.

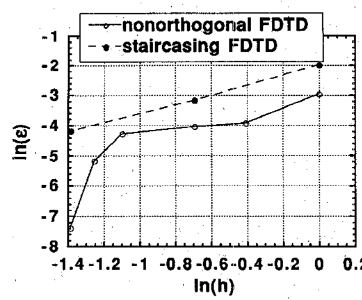

Fig. 4. A comparison of the convergence rates of the conventional FDTD method with a staircase grid and the nonorthogonal FDTD technique with a grid formed to the cylinder.

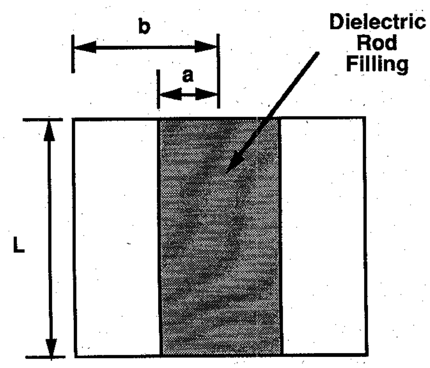

Fig. 5. Cylindrical cavity with a dielectric rod filling. (a = 1.00076 cm., b = 1.27 cm., L = 1.397 cm., ϵ_r = 37.6)

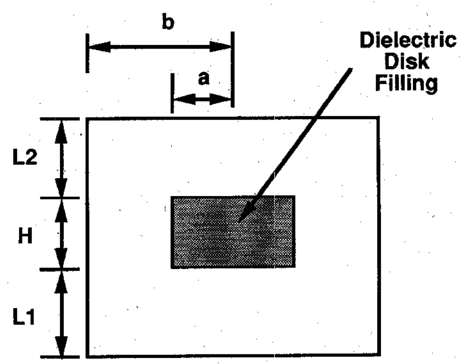

Fig. 6. Cylindrical cavity with a dielectric disk filling. (a = 0.8636 cm., b = 1.295 cm., H = 0.762 cm., $L1$ = $L2$ = 0.381 cm., ϵ_r = 35.74).

IV. Dielectric Filled Cavity Results

The lower resonant frequencies of a cylindrical cavity filled with a dielectric rod, Fig. 5, and a dielectric disk, Fig. 6, were computed and compared with results obtained from a finite element method [12] and reference [13]. The nonorthogonal-FDTD-FFT method was used to solve for the resonant frequencies in the following manner. In both cases, the mesh was chosen to conform to the shape of the cylinder and the dielectric filling, while the

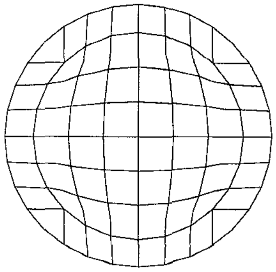

Fig. 7. Nonorthogonal grid for the cylinder with a dielectric rod filling.

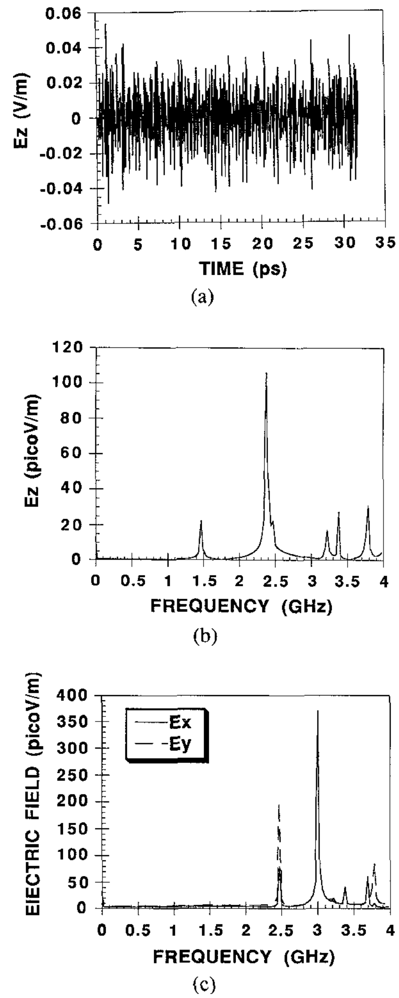

Fig. 8. (a) The time signature of an *Ez* component of the electric field at a point within the dielectric for the cylinder loaded with the dielectric rod. (b) The Fourier transform of an *Ez* field component within the dielectric for the cylinder loaded with the dielectric rod. (c) The Fourier transform of an *Ex* and *Ey* field component within the dielectric for the cylinder loaded with a dielectric rod.

mesh density was varied from a high concentration within the dielectric regions, since the majority of the fields are located there, to a low density in the air regions. A cross section of the mesh for the cylindrical cavity with the dielectric rod is shown in Fig. 7. Notice that although the mesh density is not very high, the discretized model still provides a good approximation to the cylindrical shape. It is also evident that the staircase mesh would need many more cells to achieve a similar modeling accuracy. The source for exciting the modes consisted of two loops of electric field components, viz., one in the axial and the other in the azimuthal direction, energized with the same Gaussian pulse in time. In this way, many modes could be excited while reducing the possibility of introducing fictitious non-divergence-free field components. The Gaussian pulse bandwidth was fixed at 4 GHz, providing sufficient energy over the frequency spectrum of interest. Axial and transverse electric field components were sampled at several points within the cavities to provide a means of checking the resonant frequencies. After computing the time signatures of these field components, the FFT was used to compute the frequency response from which the modes were extracted.

A time signature and the frequency responses of the electric field sampled within the cylinder filled with the dielectric rod are shown in Figures 8a through 8c. These results were computed using a chosen frequency resolution of less than 0.1 GHz and 16,384 time steps. The comparison of our results with those obtained with an FEM program, as well as those that have been published in the literature are shown in Table I. The frequency resolution of 0.1 GHz is sufficient to distinguish all but the TM011 and HE211 modes which differ by at most 0.6%. To resolve these modes, the resolution would need to be reduced to at least 0.01 GHz, requiring approximately 164 000 time steps and an extensive amount of cpu time and computer memory. Thus, the nonorthogonal-FDTD algorithm in combination with the FFT is not an efficient means of distinguishing two closely-spaced modes. Nevertheless, the resonant frequencies computed with the nonorthogonal FDTD-FFT method agree with both FEM results to within a 2.5% difference.

Table II compares the resonant frequencies obtained with the nonorthogonal FDTD-FFT technique to those obtained from [12] and a FEM method [13] for the cavity with the dielectric disk, shown in Fig. 6. The modes were extracted from the frequency spectrums given in Fig. 9(a) and (b). Considering that these results were obtained using a mesh density of 10 cells per wavelength at 3.1 GHz and that the computed resonant frequencies are above 3.4 GHz, it is surprising that the percentage difference between the nonorthogonal-FDTD results and the others is on the order of 1.5%–4.6% and not much greater. In other

TABLE I
COMPARISON OF THE LOWER ORDER RESONANT FREQUENCIES FOR THE CYLINDRICAL CAVITY WITH A DIELECTRIC ROD FILLING (ϵ_r = 37.6, a = 1.00076 cm, b = 1.27 cm, L = 1.397 cm)

Mode	Ref. [12]	FEM [13]	Nonorthogonal FDTD
	(GHz)	(GHz)	(GHz)
—	—	1.50	1.47
—	—	2.44	2.38
HE11	2.49	2.50	2.48
TM011	3.38	3.38	3.38
HE211	3.40	3.38	3.38
HE121	3.81	3.83	3.79

TABLE II
COMPARISON OF THE LOWER ORDER RESONANT FREQUENCIES FOR THE CYLINDRICAL CAVITY WITH A DIELECTRIC DISK FILLING (ϵ_r = 35.74, a = 0.8636 cm, b = 1.295 cm, H = 0.762 cm, $L1$ = $L2$ = 0.381 cm)

Mode	Ref. [12] (GHz)	FEM [13] (GHz)	Nonorthogonal FDTD (GHz)	% Difference (Ref. [12] & FDTD)
TE01	3.44	3.51	3.53	2.9%
HE11	4.27	4.27	4.17	2.3%
HE12	4.37	4.36	4.53	4.6%
TM01	4.60	4.54	4.62	1.5%

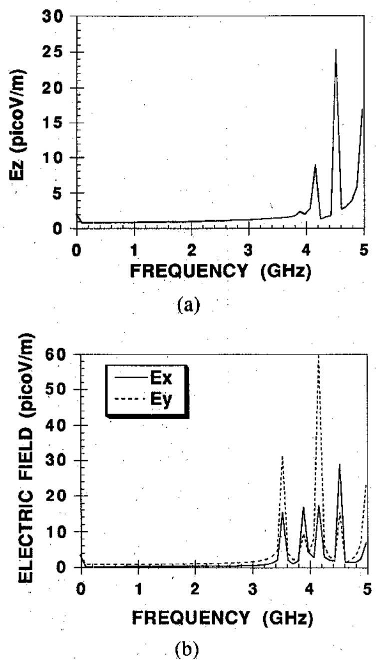

Fig. 9. (a) The Fourier transform of an *Ez* field component within the dielectric for the cylindrical cavity loaded with the dielectric disk. (b) The Fourier transform of an *Ex* and *Ey* field component within the dielectric for the cylindrical cavity loaded with the dielectric disk.

words, even with a sparse mesh density, the nonorthogonal-FDTD technique in combination with the FFT produces results in reasonable agreement with those of the two different FEM methods for the dielectric-button filled cylindrical cavity.

We have demonstrated that the nonorthogonal FDTD-FFT combination can be used to compute the lower order resonant frequencies of dielectric filled cylinders. It is relatively straight forward to extend the procedure to cavities with arbitrary fillings and shapes, because one needs only to reform the mesh and specify the material properties. It should be pointed out that there are some limitations in using the time-domain technique for computing very high-order modes, since one needs to employ a very dense mesh in order to handle the small wavelengths. This, in turn, requires extensive computer memory and, since the associated time step must also be small, a large amount of computer time as well. Also, for modes that are in close proximity to each other, many time steps may be required to achieve the frequency resolution necessary for distinguishing these modes with the FFT. A surprising conclusion of the analysis of the cavity with the dielectric disk is that the resonant frequencies computed with the nonorthogonal FDTD-FFT combination are in good agreement with both of the FEM results even though the nonorthogonal-mesh density is less than 10 cells per wavelength at the frequencies of interest. More rigorous analysis however is needed before a nonorthogonal-mesh-density criteria can be formulated for complicated geometries.

V. CONCLUSION

In this paper, the nonorthogonal-FDTD technique was extended to the problem of computing the resonant frequencies of dielectric-filled cylindrical cavities that can be of arbitrary shape and have arbitrary fillings. The results of a convergence study were presented to demonstrate that, for cylindrical structures, a significant improvement in the rate of convergence of the nonorthogonal-FDTD technique can be achieved over the conventional FDTD method. In fact, the results of the convergence study lead us to conclude that this nonorthogonal approach will always converge faster than the conventional FDTD for structures with arbitrary shape, provided the nonorthogonal mesh is free of singularity problems, and the cells are not overly distored.

As a demonstration of the technique, the lower resonant frequencies of two dielectric-filled cylindrical cavities were computed with the nonorthogonal-FDTD technique and compared with the resonant frequencies obtained from two different finite element techniques. The comparisons showed the results of the FDTD technique agreed reasonably well with those of the finite element methods, even when a sparse mesh, viz., less than 10 cells per wavelength, was used with the FDTD. It is anticipated that the nonorthogonal-FDTD results can be improved by using signal processing techniques other than the FFT. Ideally, the chosen technique should be able to extract the modes from the fields computed over a relatively short time interval as compared with that required for the FFT. Currently, Proney's method is being investigated as a possible alternative to the FFT for improving frequency resolution and reducing the computational time [14].

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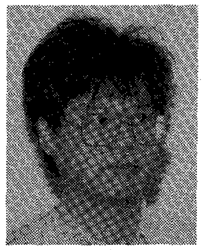
Paul H. Harms (M'91) received the B.E.E. degree in electrical engineering from the Georgia Institute of Technology in 1984 and the M.S. degree in electrical engineering from the University of Illinois in 1987. He is currently pursuing the Ph.D. degree in electrical engineering at the University of Illinois.

His present interests are in the numerical and experimental electromagnetic analysis of passive microwave components and electronic packaging for high-speed digital circuits.



Jin-Fa Lee (M'88) was born in Taipei, Taiwan, in 1960. He received the B.S. degree from National Taiwan University, in 1982 and the M.S. and Ph.D. degrees from Carnegie Mellon University in 1986 and 1989, respectively, all in electrical engineering.

From 1988 to 1990, he was with ANSOFT Corp., where he developed several CAD/CAE finite element programs for modeling three-dimensional microwave and millimeter-wave circuits. From 1990 to 1991, he was Post-Doctoral Fellow at the Electromagnetic Communication Laboratory, University of Illinois at Urbana-Champaign. Currently, he is an Assistant Professor at Department of Electrical Engineering, Worcester Polytechnic Institute. Dr. Lee's current research interests are analyses of numerical methods, couplings of active and passive components in the high-speed electronic circuits, solution of the moving boundary problem and its application to semiconductor process modelings, and EM field propagation in linear and/or non-linear medium.

Raj Mittra (S'54–M'57–SM'69–F'71) is Director of the Electromagnetic Communication Laboratory of the Electrical and Computer Engineering Department and Research Professor of the Coordinated Science Laboratory at the University of Illinois. He is a former president of AP-S, and he has served as the editor of the IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION. He is president of RM Associates, a consulting organization providing services to several industrial and governmental organizations.

Dr. Mittra's professional interests include the areas of analytical and computer-aided electromagnetics, high-speed digital circuits, radar scattering, satellite antennas, microwave and millimeter-wave integrated circuits, frequency selective surfaces, EMP and EMC analysis, and remote sensing.